First-order substructural logics

So different, yet so familiar

Petr Cintula

Institute of Computer Science of the Czech Academy of Sciences
cintula@cs.cas.cz

Substructural logics are a prominent family of non-classical logics whose complete semantics is given by classes of residuated lattices. While there is a vast literature on all thinkable aspects of *propositional* substructural logics (see e.g. monographs [4, 10, 13] and the references therein), the works focusing on *first-order predicate* substructural logics are few and far between (with most prominent contributions by Hiroakira Ono, see e.g. [9]).

Classical predicate logic interprets n-ary predicates as mappings from the n-th power of a given domain into the two-valued Boolean algebra **2**. The idea of replacing **2** by a more general structure is very natural and was shown to lead to very interesting results: prime examples are the Boolean-valued or Heyting-valued models of set theory [1] (or even more general models proposed e.g. by Takeuti and Titani [14] or by Hájek and Haniková [6]). There exists a stream of research, by Mostowski [8], Rasiowa [11], Sikorski [12], Horn [7], and Hájek [5] to give just a few names, studying logics where predicates can take values in lattices (with operators) from a certain class.

This talk presents a general framework for studying predicate substructural logics (based on even more general framework developed in cooperation with Carles Noguera [3]). Given a substructural logic L, we add the usual axioms of classical/intuitionistic logic to obtain predicate logic $\mathrm{L}\forall^m$ and easily prove its completeness w.r.t. models built over algebras from $\mathbf{Alg}^*(\mathrm{L})$ (the class of residuated lattices providing the equivalent algebraic semantic of L). It is easy to see that we can restrict to complete lattices from $\mathbf{Alg}^*(\mathrm{L})$ if (but not only if) $\mathbf{Alg}^*(\mathrm{L})$ admits Dedekind–MacNeille completions.

In classical logic we can however improve this completeness theorem much more: we can restrict to models built over the algebra **2** only. Analogous result for substructural logics would be the completeness w.r.t. models over algebras which are (finitely) subdirectly irreducible relative to $\mathbf{Alg}^*(\mathrm{L})$. This however is rarely the case. We show which axioms/rules has to be added to $\mathrm{L}\forall^m$ to obtain this completeness result for a rather wide family of substructural logics.

The core of our proof is an appropriate generalization of the classical *Henkin method*. We present its modular version which allows to prove not only the desired completeness theorem, but also certain (natural) form of *Skolemization* and certain (perhaps not so natural) form of the *omitting type theorem* (a joint work with Denisa Diaconescu [2]).

Throughout the talk we will see that sometimes we have to twist the classical notions quite a lot to obtain the analogs of the classical results. To show that it is not always so bad we conclude the talk by presenting the (almost) classical proof of the fact that already a very weak variant of Robinson arithmetic is *essentially undecidable* when formulated in any substructural logic which still obeys the structural rule of weakening (a work in progress with Guillermo Badia and Andrew Tedder).

References

[1] John L. Bell. *Set Theory: Boolean-Valued Models and Independence Proofs*, volume 47 of *Oxford Logic Guides*. Oxford University Press, 2 edition, 2011.

[2] Petr Cintula and Denisa Diaconescu. Omitting types theorem for fuzzy logics. *IEEE Transactions on Fuzzy Systems*, 27(2):273–277, 2019.

[3] Petr Cintula and Carles Noguera. A Henkin-style proof of completeness for first-order algebraizable logics. *Journal of Symbolic Logic*, 80(1):341–358, 2015.

[4] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[5] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.

[6] Petr Hájek and Zuzana Haniková. A development of set theory in fuzzy logic. In Melvin Chris Fitting and Ewa Orlowska, editors, *Beyond Two: Theory and Applications of Multiple-Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, pages 273–285. Springer, Heidelberg, 2003.

[7] Alfred Horn. Logic with truth values in a linearly ordered Heyting algebras. *Journal of Symbolic Logic*, 34(3):395–408, 1969.

[8] Andrzej Mostowski. Axiomatizability of some many valued predicate calculi. *Fundamenta Mathematicae*, 50(2):165–190, 1961.

[9] Hiroakira Ono. Algebraic semantics for predicate logics and their completeness. In Ewa Orlowska, editor, *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*, pages 637–650. Physica-Verlag, Heidelberg, 1999.

[10] Francesco Paoli. *Substructural Logics: A Primer*, volume 13 of *Trends in Logic*. Kluwer, Dordrecht, 2002.

[11] Helena Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, Amsterdam, 1974.

[12] Helena Rasiowa and Roman Sikorski. *The Mathematics of Metamathematics*. Panstwowe Wydawnictwo Naukowe, Warsaw, 1963.

[13] Greg Restall. *An Introduction to Substructural Logics*. Routledge, New York, 2000.

[14] Gaisi Takeuti and Satoko Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *Journal of Symbolic Logic*, 49(3):851–866, 1984.